INVERSION OF NEPHELOMETER MEASUREMENTS TO INFER RADIAL MOMENTS OF THE AEROSOL SIZE DISTRIBUTION

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Can measurements routinely made with a nephelometer be used to determine the lowest several moments of the aerosol size distribution?

Assumptions:

- Spherical particles of known index of refraction
- Ideal nephelometer with no angular truncation
- Backscatter fraction $BSF = (1 - g)/2$

Required - knowledge of:

- Optical properties of aerosol particles
- Moments of aerosol size distributions
- Inversion of nephelometer measurements

OPTICAL PROPERTIES OF AEROSOL PARTICLES

Extinction cross section $\sigma_{ep} = \pi r^2 Q_{\text{ext}}$

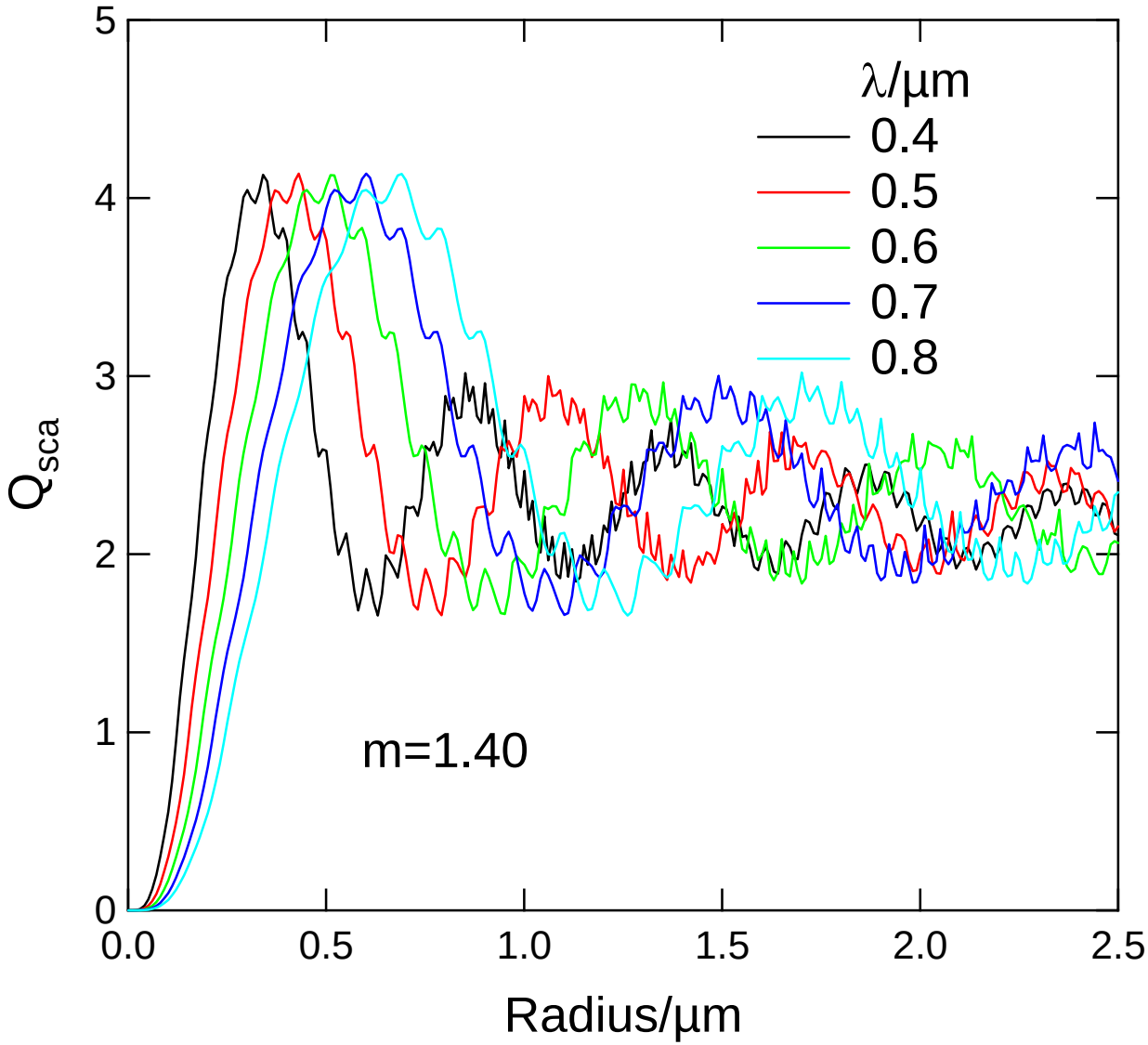
Scattering cross section $\sigma_{sp} = \pi r^2 Q_{\text{sca}}$

Single-scattering albedo $\omega = \dfrac{Q_{\text{sca}}}{Q_{\text{ext}}}$

Backscatter fraction $BSF = \int_{\pi/2}^{\pi} \sin\theta d\theta \int_{0}^{2\pi} d\phi \left[F(\theta,\phi) \right]$

Asymmetry parameter $g = \int_{0}^{\pi} \sin\theta d\theta \int_{0}^{2\pi} d\phi \left[\cos\theta \cdot F(\theta,\phi) \right]$

SCATTERING EFFICIENCY Q_{sca}



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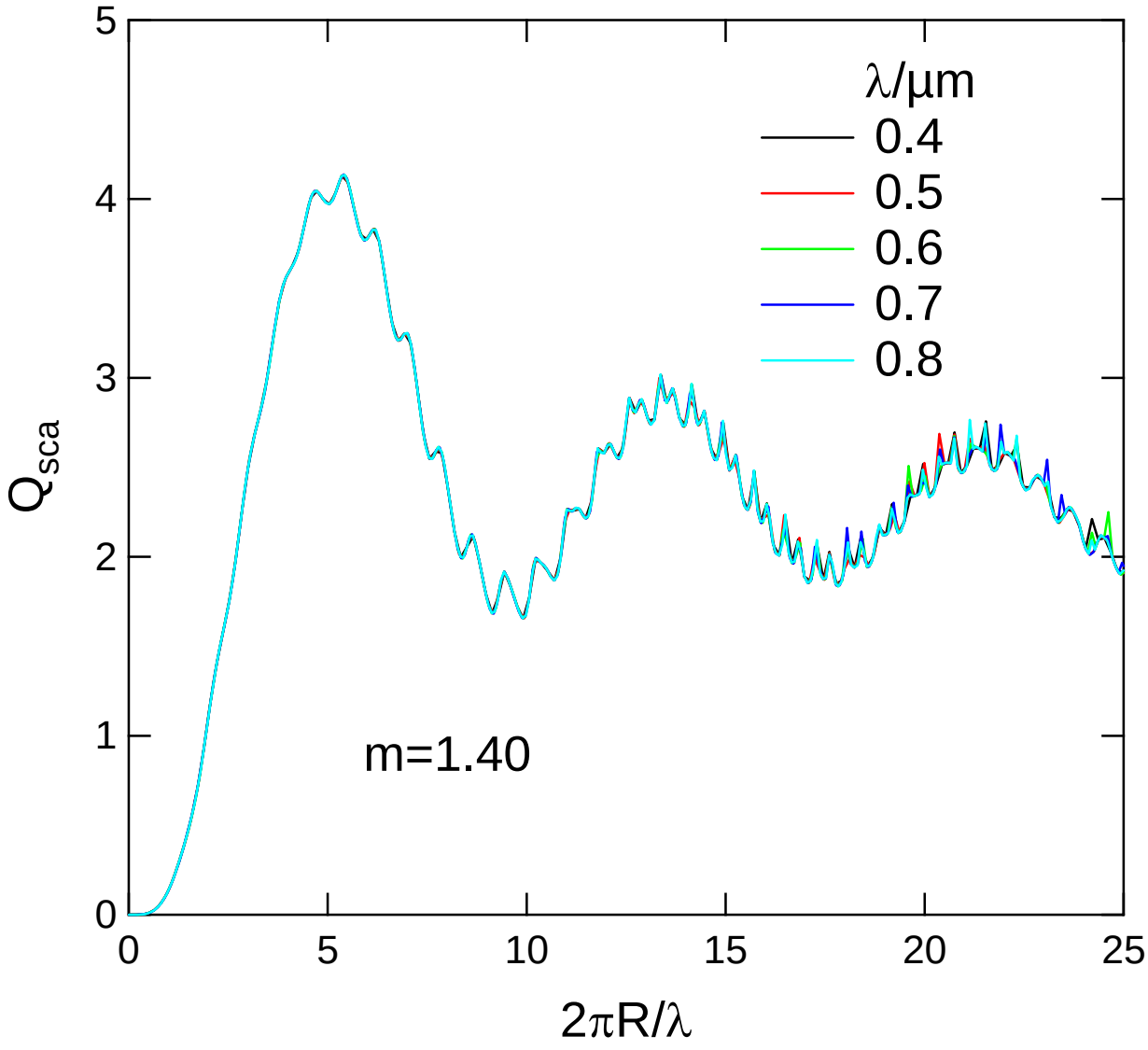

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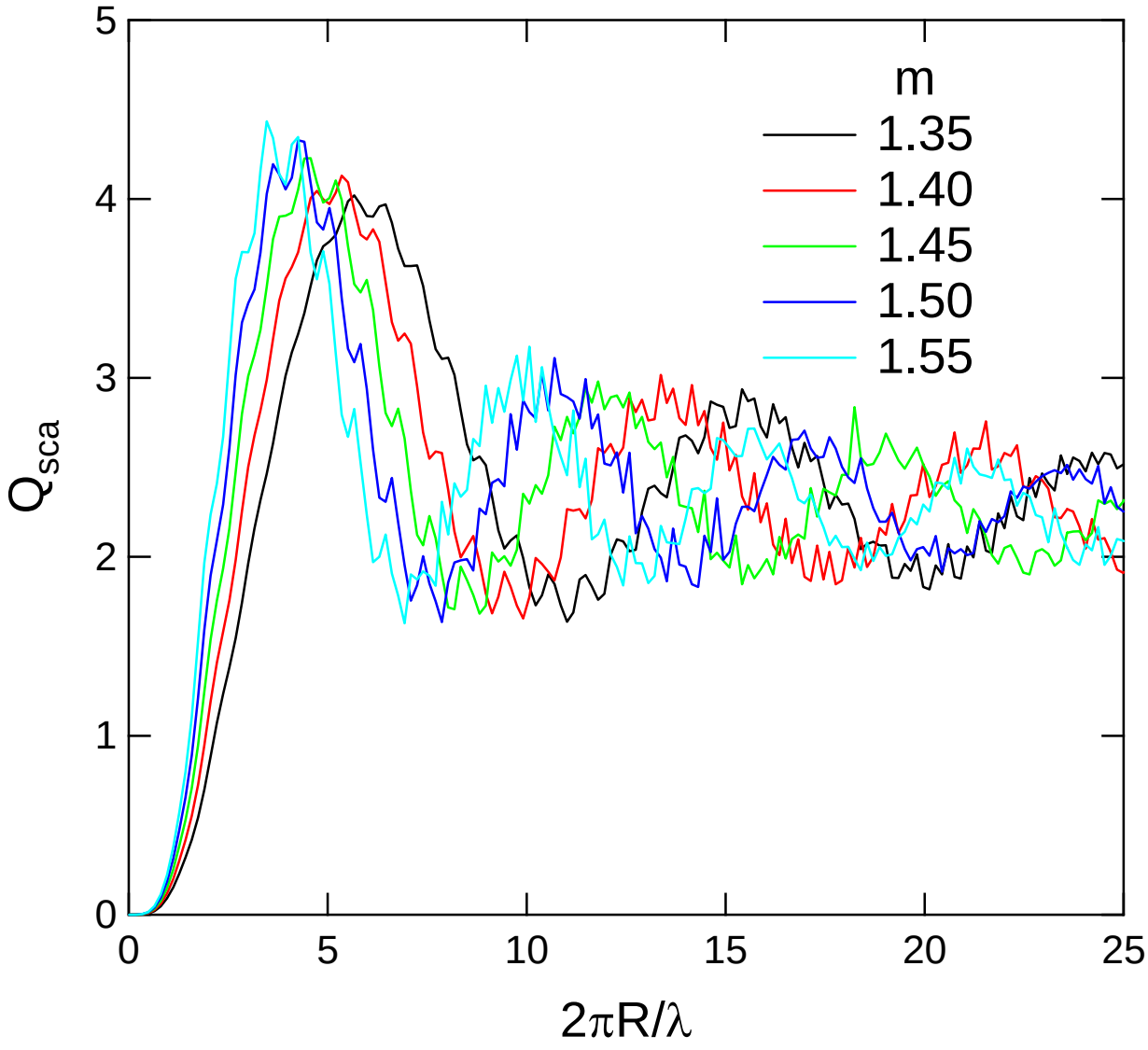

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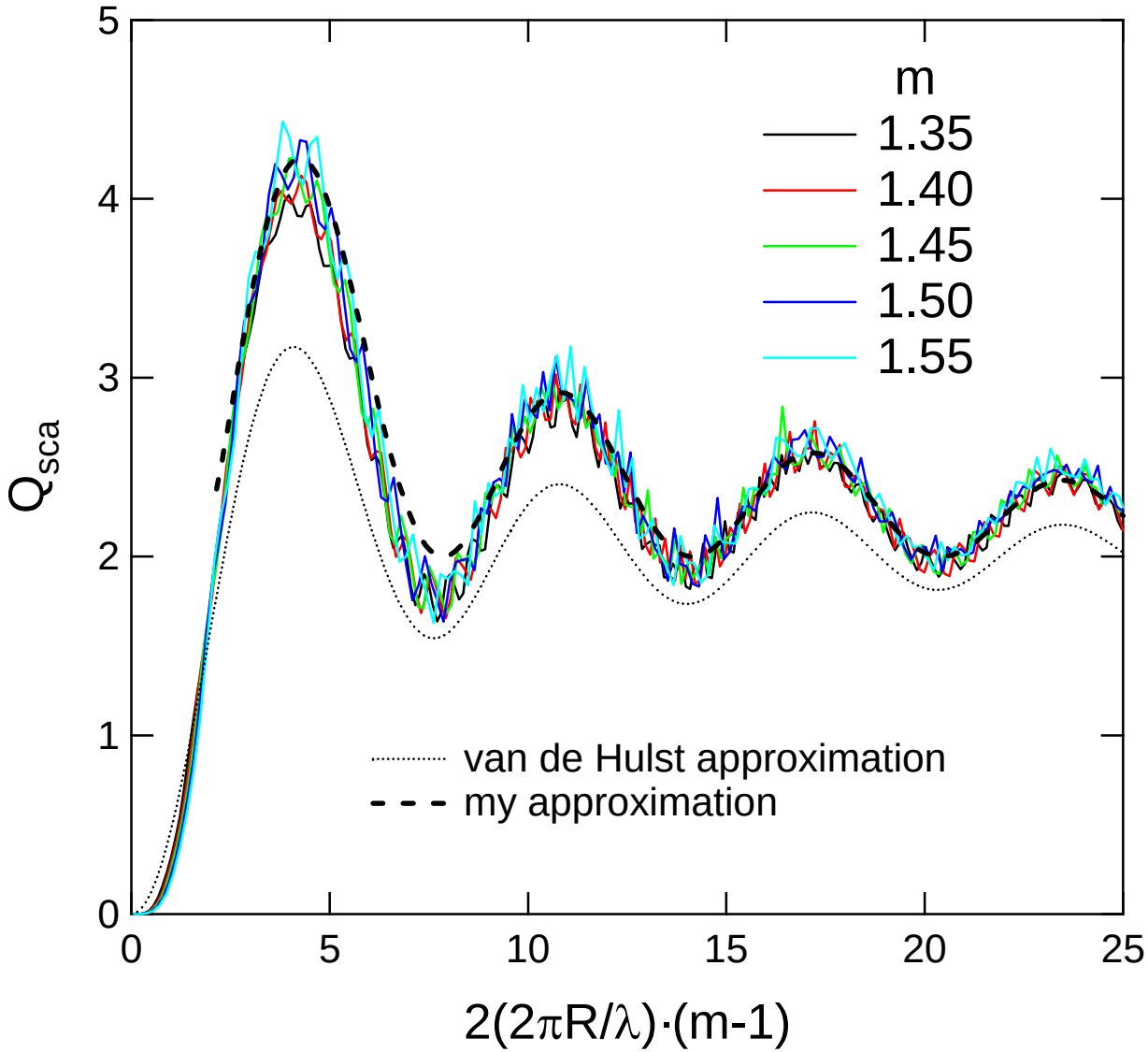
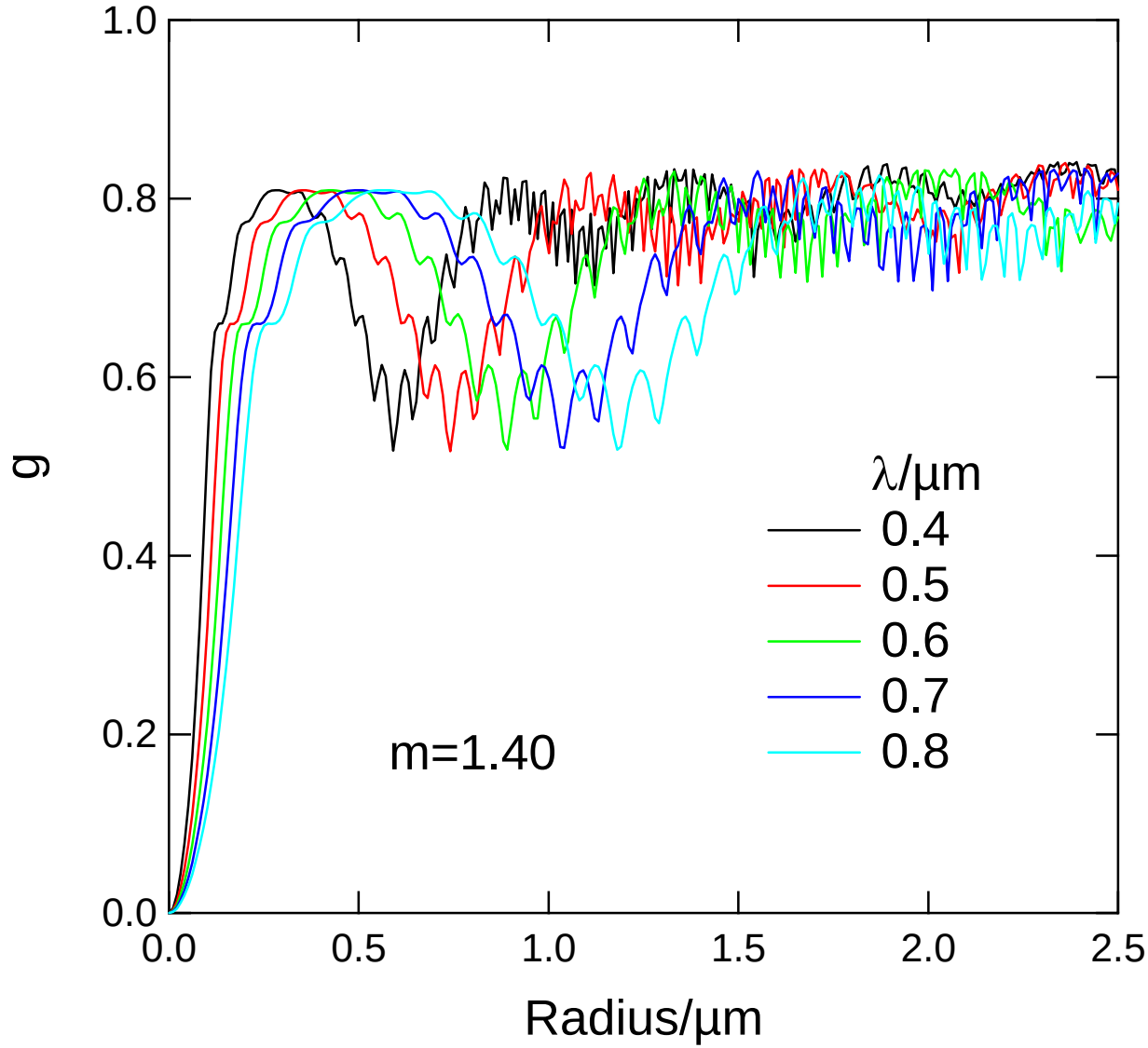


ASYMMETRY PARAMETER *g*



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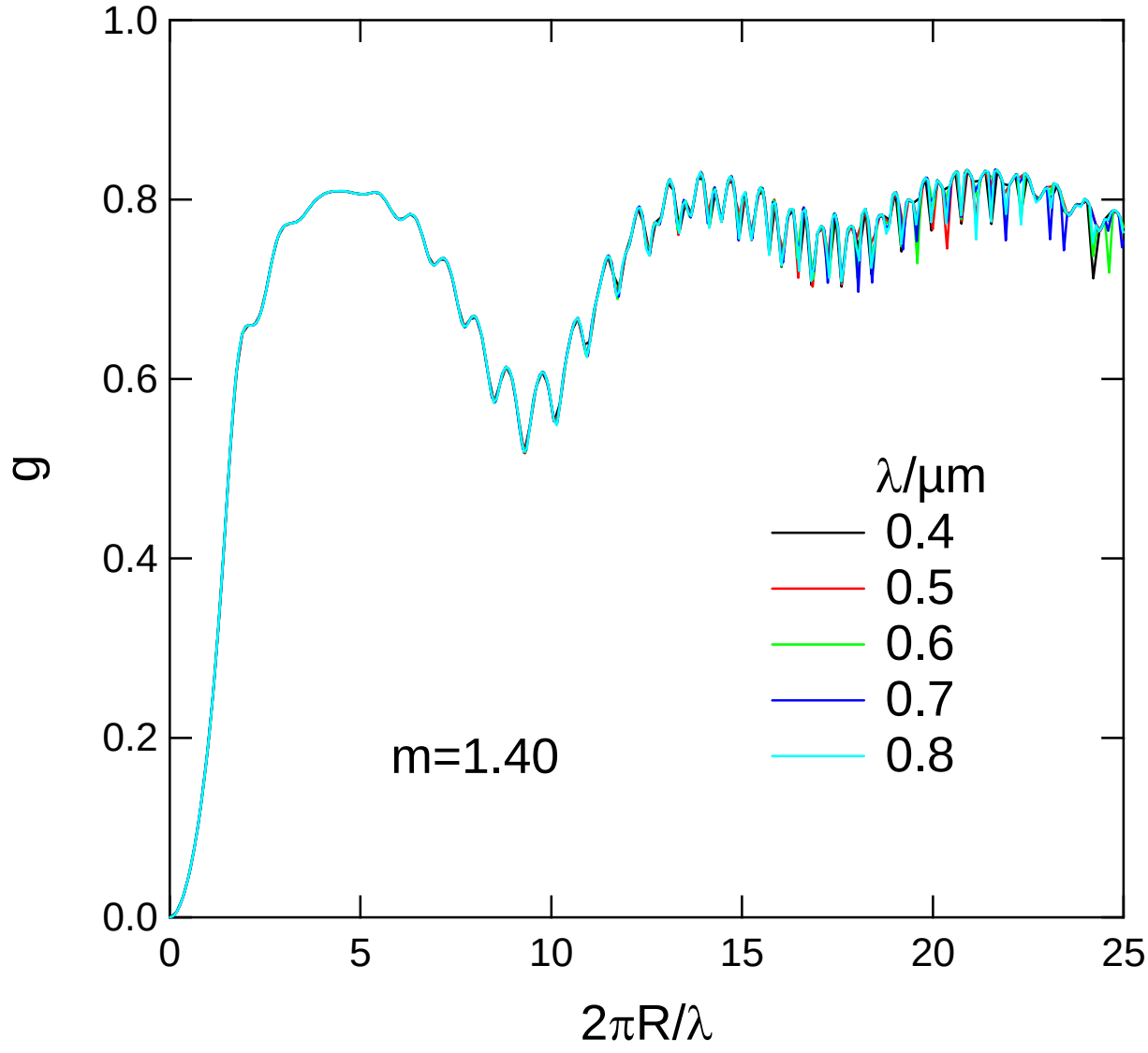

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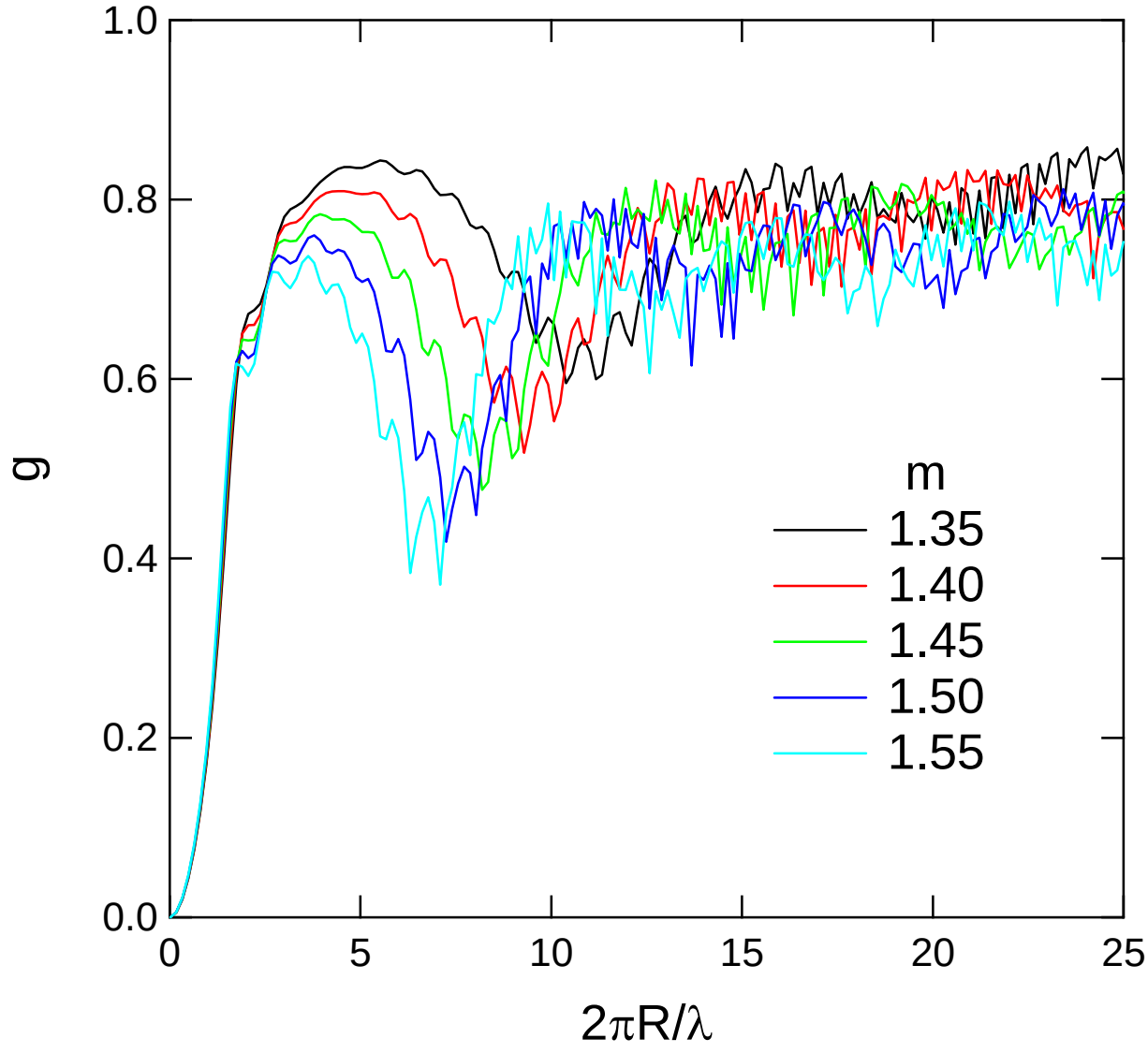

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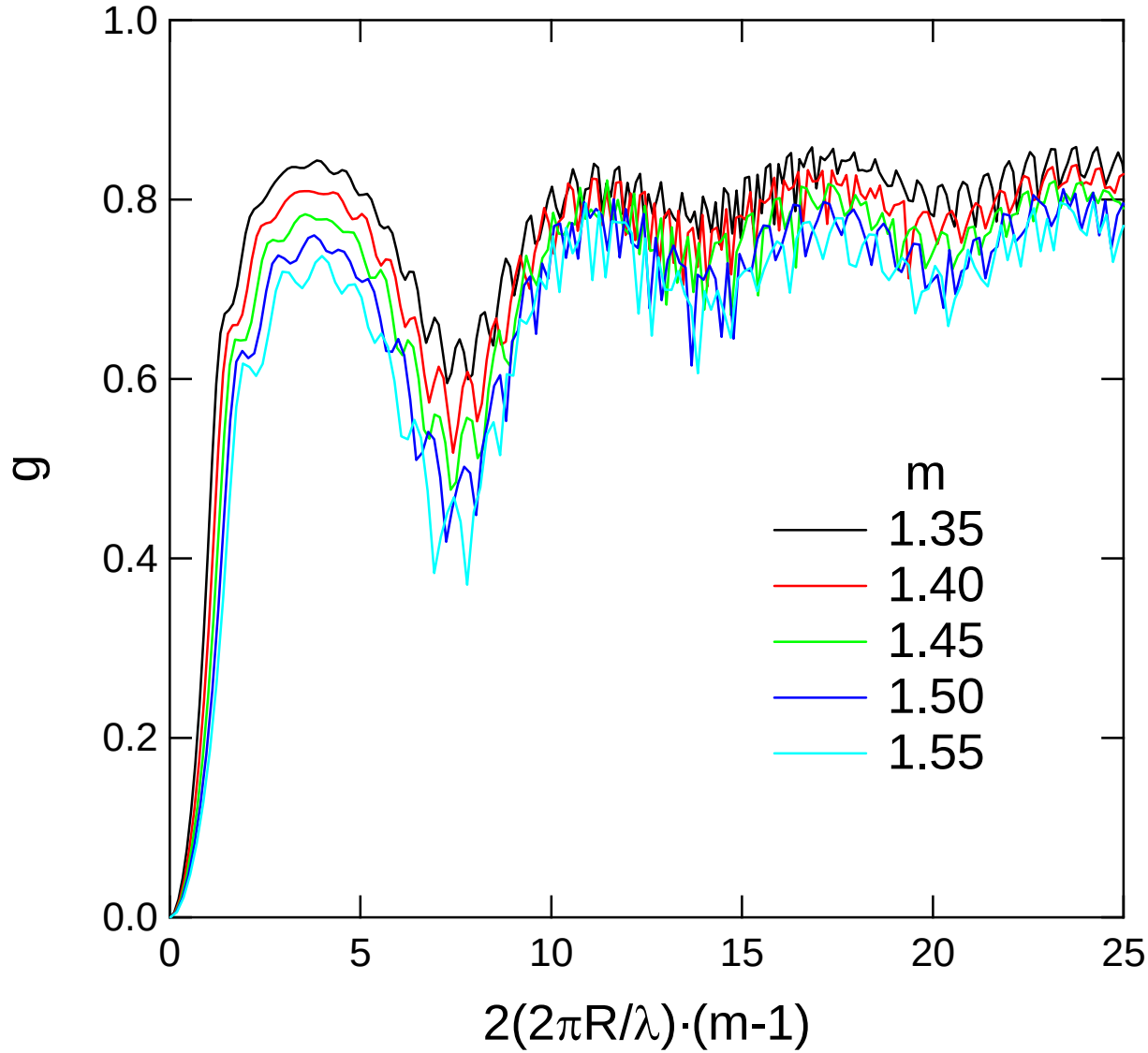

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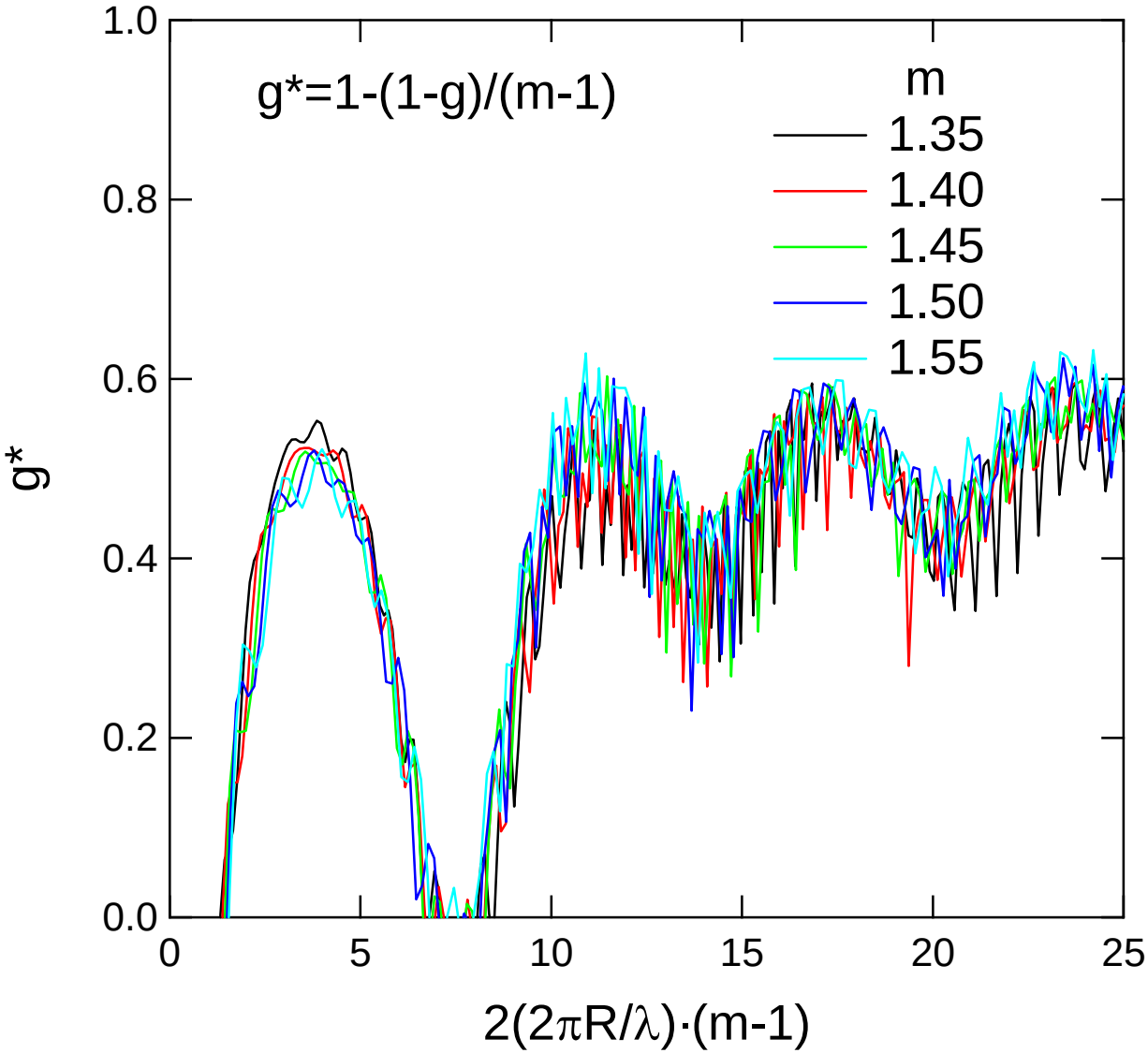
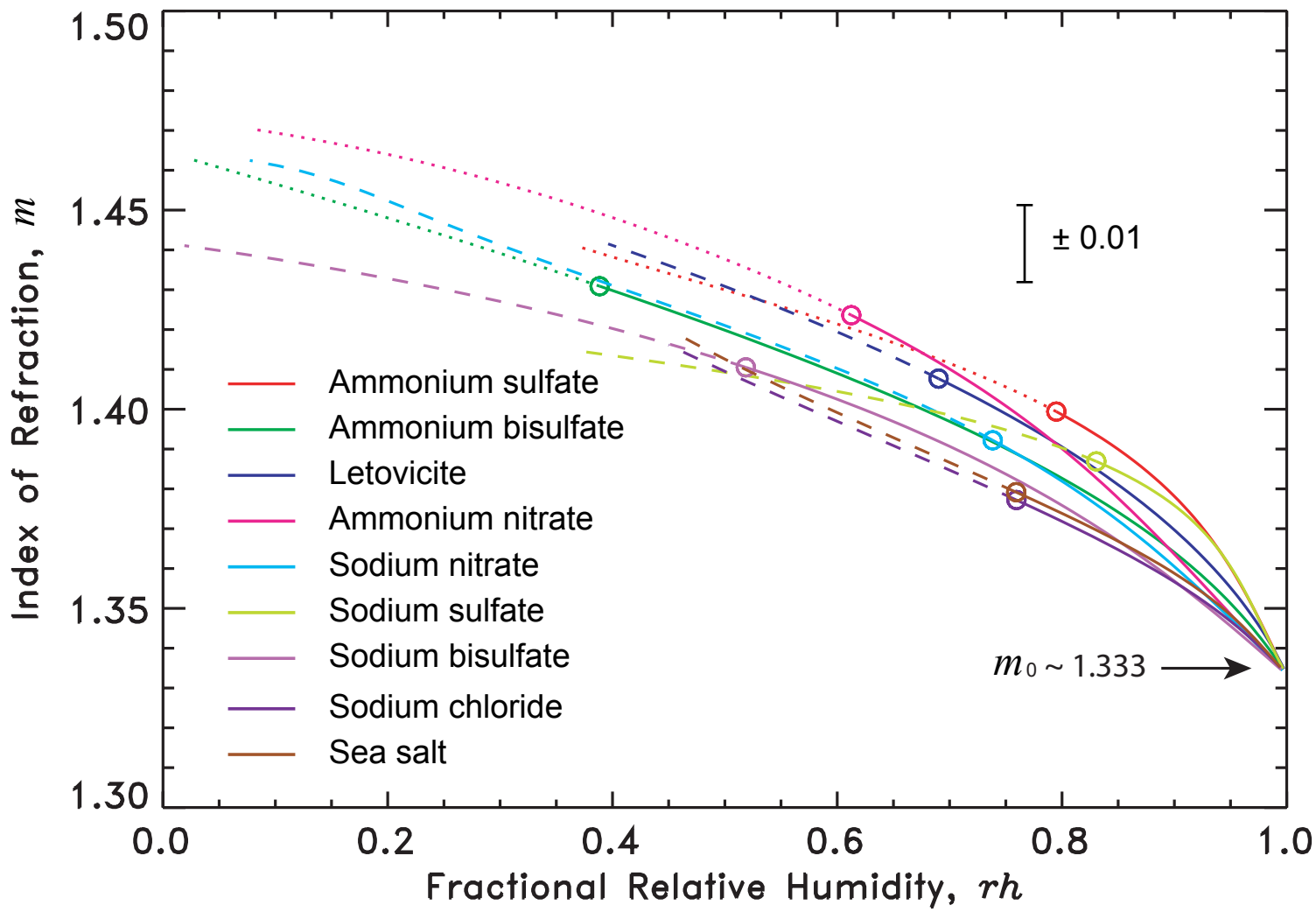
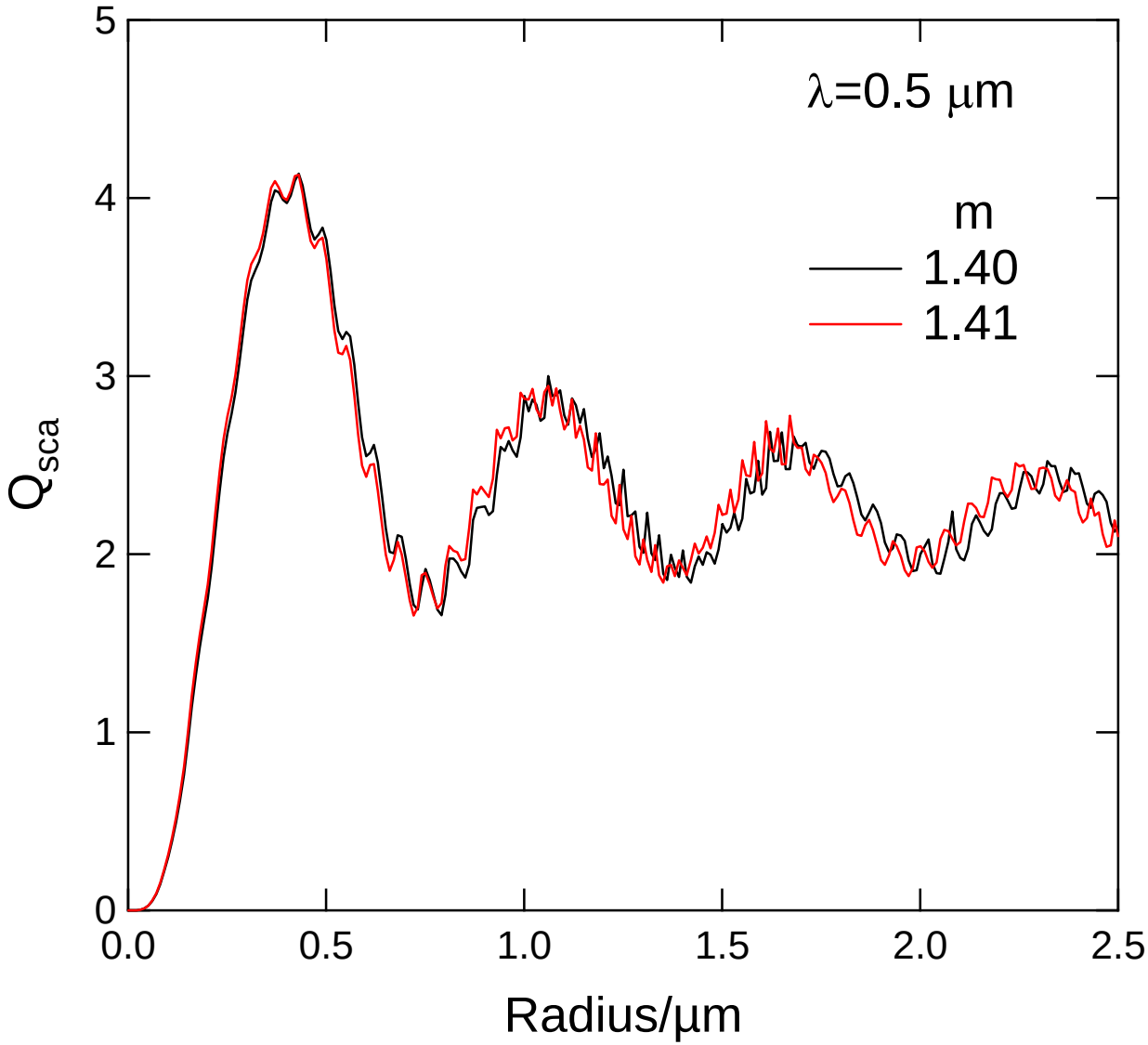


Dependence of Index of Refraction on RH

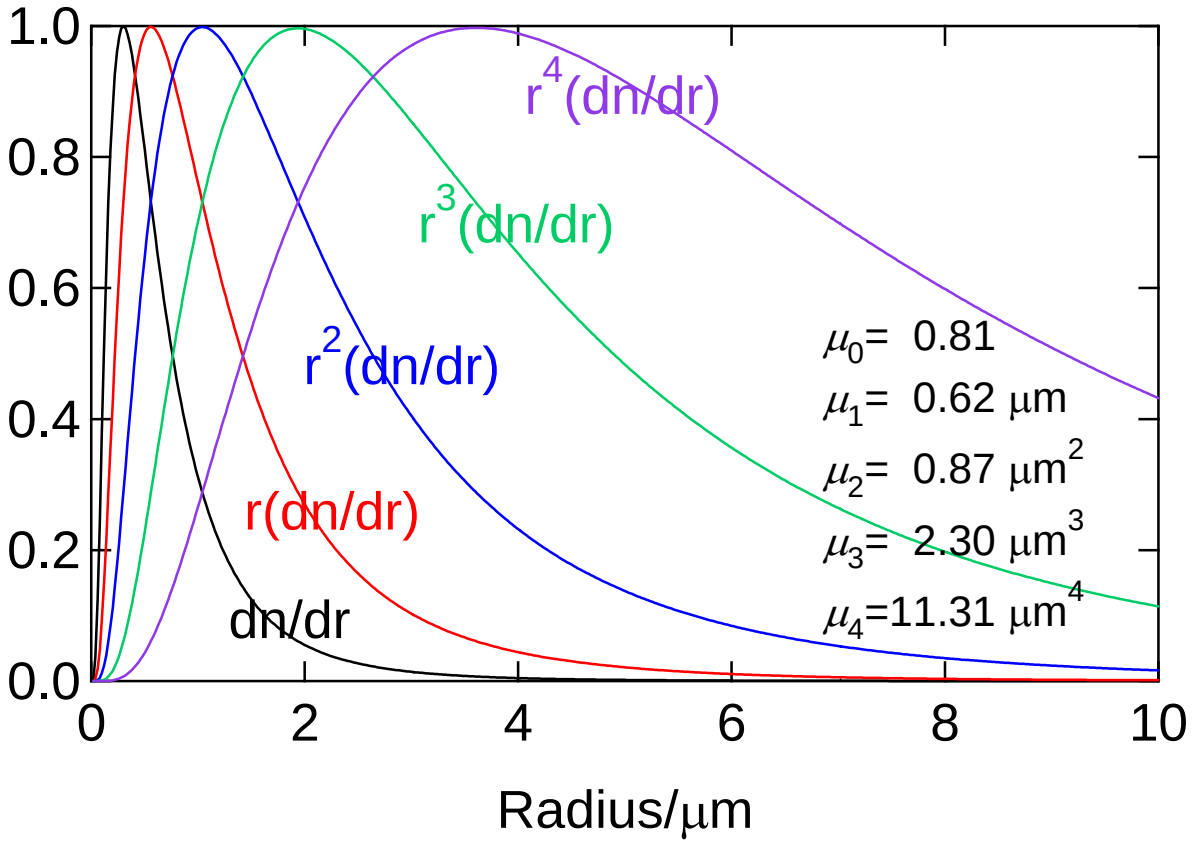


SCATTERING EFFICIENCY Q_{sca}



RADIAL MOMENTS OF AN AEROSOL SIZE DISTRIBUTION

$$\mu_p = \int r^p \frac{dn}{dr} dr$$



QUANTITIES DEFINED BY MOMENTS

Total quantities:

Number concentration: $N_{\text{tot}} = \mu_0$

Area concentration: $A_{\text{tot}} = 4\pi\mu_2$

Volume concentration: $V_{\text{tot}} = \frac{4\pi}{3}\mu_3$

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"Effective" quantities:

Radius: $r_{\text{eff}} = \frac{\mu_3}{\mu_2}$

Variance: $v_{\text{eff}} = \frac{\mu_4\mu_2}{\mu_3^{\,2}} - 1$

Skewness: $s_{\text{eff}} = \mu_2 \left[\frac{\mu_5\mu_2^{\,2} - 3\mu_4\mu_3\mu_2 + 2\mu_3^{\,3}}{\left(\mu_4\mu_2 - \mu_3^{\,2}\right)^{3/2}} \right]$

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<u>The lowest several moments contain essential information about dn/dr.</u>

NEPHELOMETER MEASUREMENTS*

$$\phi_1 = \int Q_{\text{sca}}\left(450\,\text{nm}, r\right)\left[\pi r^2 \frac{dn}{dr}\right] dr$$

* <u>*idealized*</u>:- $BSA \approx Q_{\text{sca}}(1-g)/2$, no cutoff angles, etc.

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$$\phi_1 = \int Q_{\text{sca}}\left(450\,\text{nm}, r\right)\left[\pi r^2 \frac{dn}{dr}\right] dr \qquad = \int K_1\left(r\right)\frac{dn}{dr}\,dr$$

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$$\phi_1 = \int Q_{\text{sca}}(450\,\text{nm}, r)\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_1(r)\frac{dn}{dr}\,dr$$

$$\phi_2 = \int Q_{\text{sca}}(550\,\text{nm}, r)\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_2(r)\frac{dn}{dr}\,dr$$

$$\phi_3 = \int Q_{\text{sca}}(700\,\text{nm}, r)\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_3(r)\frac{dn}{dr}\,dr$$

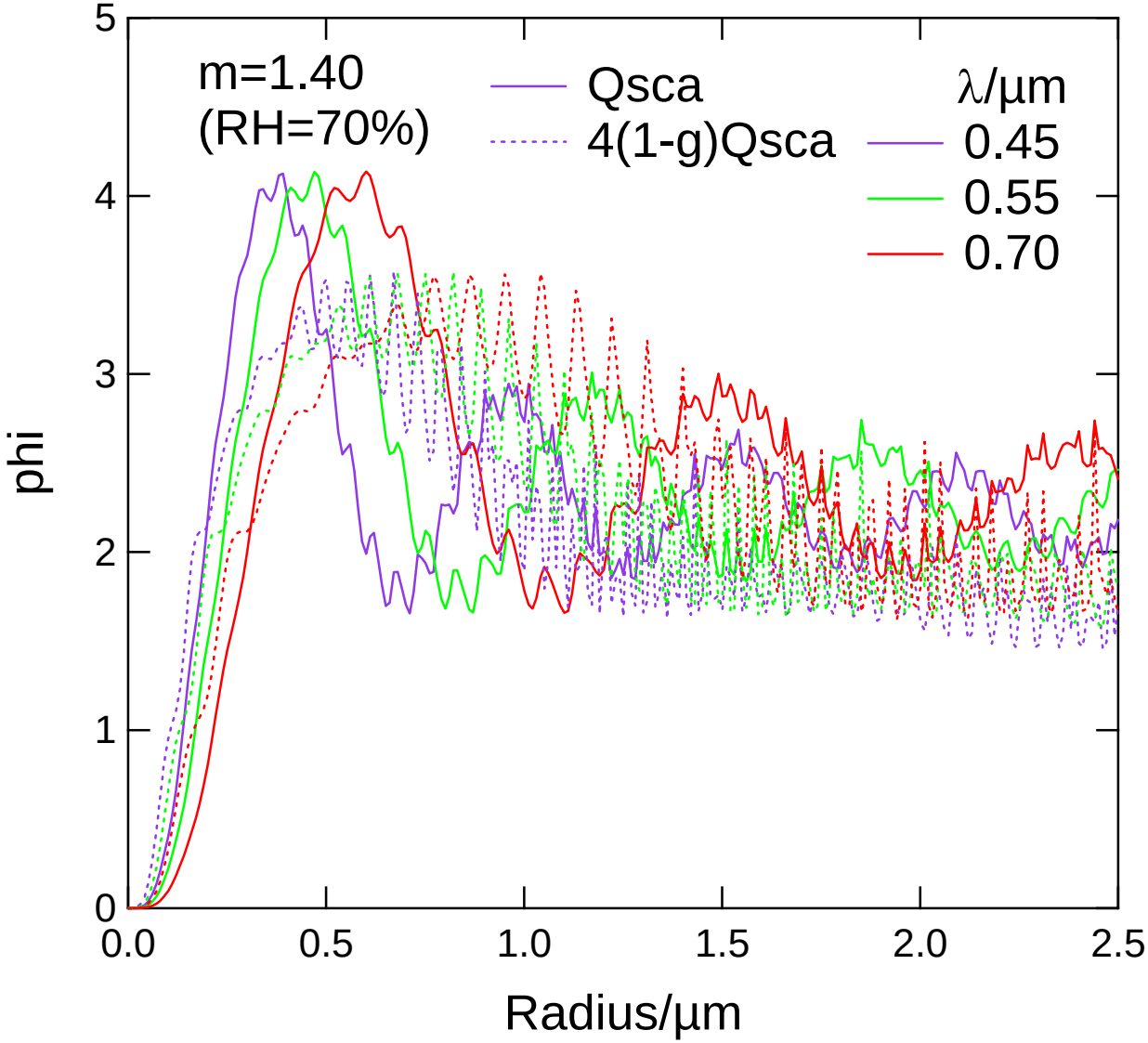
$$\phi_4 = \int Q_{\text{sca}}(450\,\text{nm}, r)\left[1 - g(450\,\text{nm}, r)\right]\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_4(r)\frac{dn}{dr}\,dr$$

$$\phi_5 = \int Q_{\text{sca}}(550\,\text{nm}, r)\left[1 - g(550\,\text{nm}, r)\right]\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_5(r)\frac{dn}{dr}\,dr$$

$$\phi_6 = \int Q_{\text{sca}}(700\,\text{nm}, r)\left[1 - g(700\,\text{nm}, r)\right]\left[\pi r^2 \frac{dn}{dr}\right] dr = \int K_6(r)\frac{dn}{dr}\,dr$$

* *idealized*:- $BSA \approx Q_{\text{sca}}(1-g)/2$, no cutoff angles, etc.

Q_{sca}, *BSF* AS FUNCTIONS OF RADIUS



DETERMINATION OF MOMENTS: THE INVERSION PROCESS

$$\phi_i = \int K_i(r) \frac{dn}{dr} dr$$

1) Represent the lowest several powers of radius as linear combinations of *K* quantities.

The *K* quantities serve as a basis set; they are NOT orthogonal!

<u>*This is the hard part.*</u>

2) Substitute into the definitions of the moments.

This is easy.

3) Collect terms.

The lowest several moments are expressed as linear combinations of the measured ϕ quantities.

THE MATHEMATICS OF INVERSION

$$r^0 = a_{01}K_1 + a_{02}K_2 + a_{03}K_3 + \ldots$$

$$r^1 = a_{11}K_1 + a_{12}K_2 + a_{13}K_3 + \ldots$$

1) $\ldots$

$$r^p = a_{p1}K_1 + a_{p2}K_2 + a_{p3}K_3 + \ldots$$

$\ldots$

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$$\begin{pmatrix} r^p \end{pmatrix} = \left(A_{pj}\right)\begin{pmatrix} K \end{pmatrix}$$

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2) $$\mu_p = \int r^p \frac{dn}{dr} dr$$

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$$\mu_p = \int r^p \frac{dn}{dr} dr$$

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$\ldots$

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2) $$= \int \left[a_{p1}K_1 + a_{p2}K_2 + a_{p3}K_3 + \ldots\right] \frac{dn}{dr} dr$$

$$= a_{p1}\left[\int K_1 \frac{dn}{dr} dr\right] + a_{p2}\left[\int K_2 \frac{dn}{dr} dr\right] + \ldots$$

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$$\begin{pmatrix} \mu_p \end{pmatrix} = \left(A_{pj}\right)\begin{pmatrix} \phi \end{pmatrix}$$

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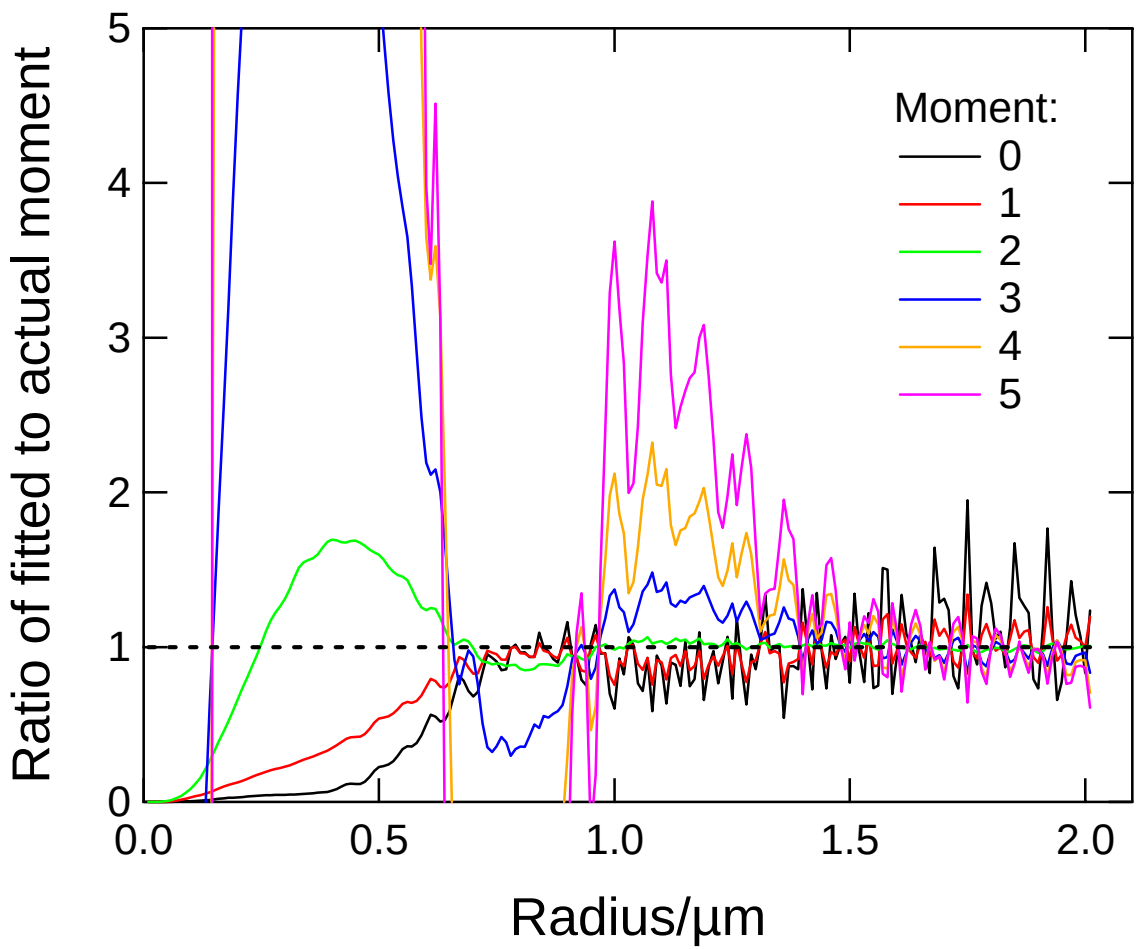
$$\begin{pmatrix} \mu_p \end{pmatrix} = \left(A_{pj}\right)\begin{pmatrix} \phi \end{pmatrix}$$

3) $$\underline{\mu_p = a_{p1}\phi_1 + a_{p2}\phi_2 + a_{p3}\phi_3 + \ldots}$$

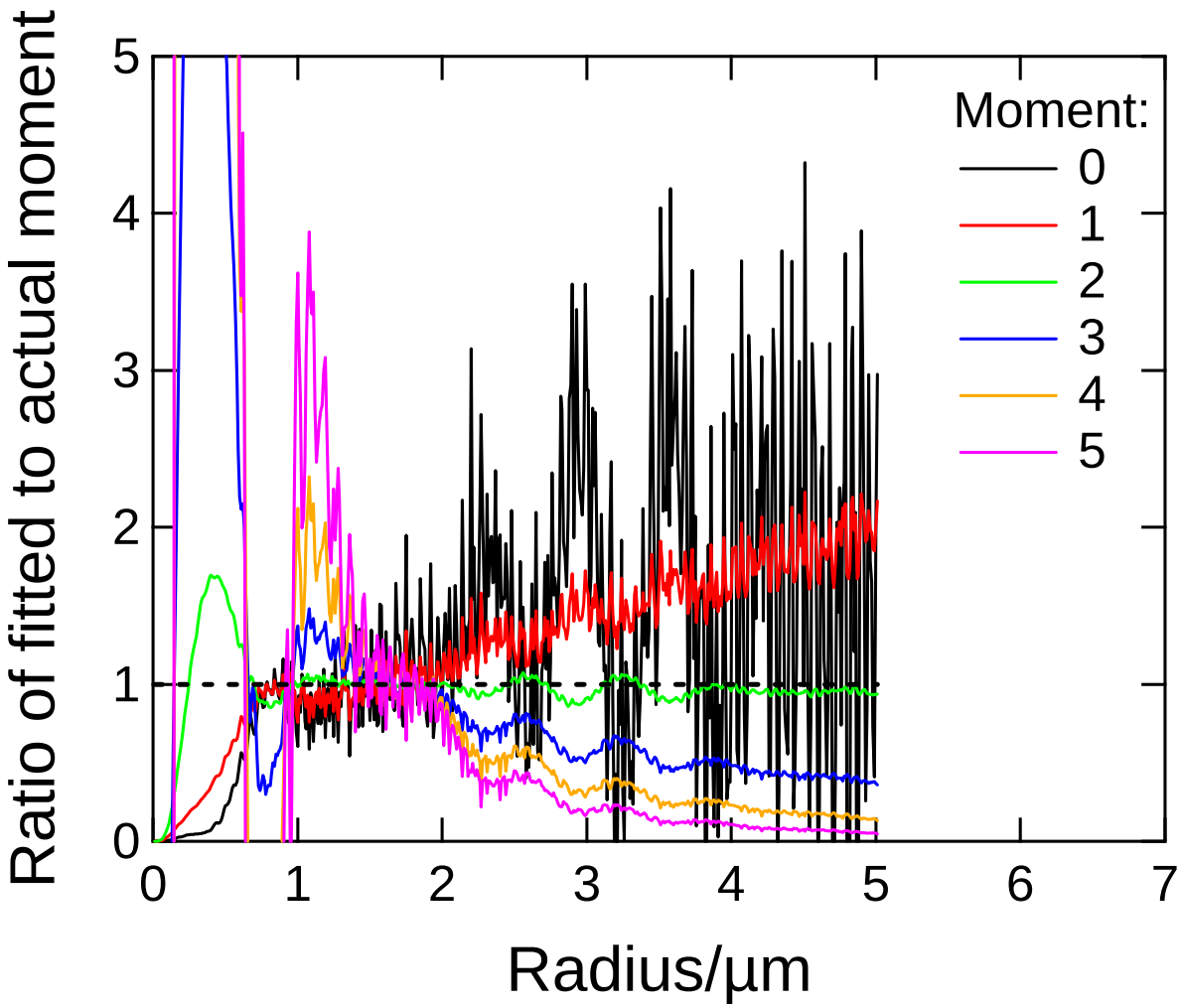
HOW WELL DOES IT WORK?

$r < 2$ μm, $m = 1.40$ (~ 70% RH), λ = 450, 550, 700 μm



RESULTS DEPEND ON SIZE RANGE

Extend reconstructed functions to $r = 5$ μm:



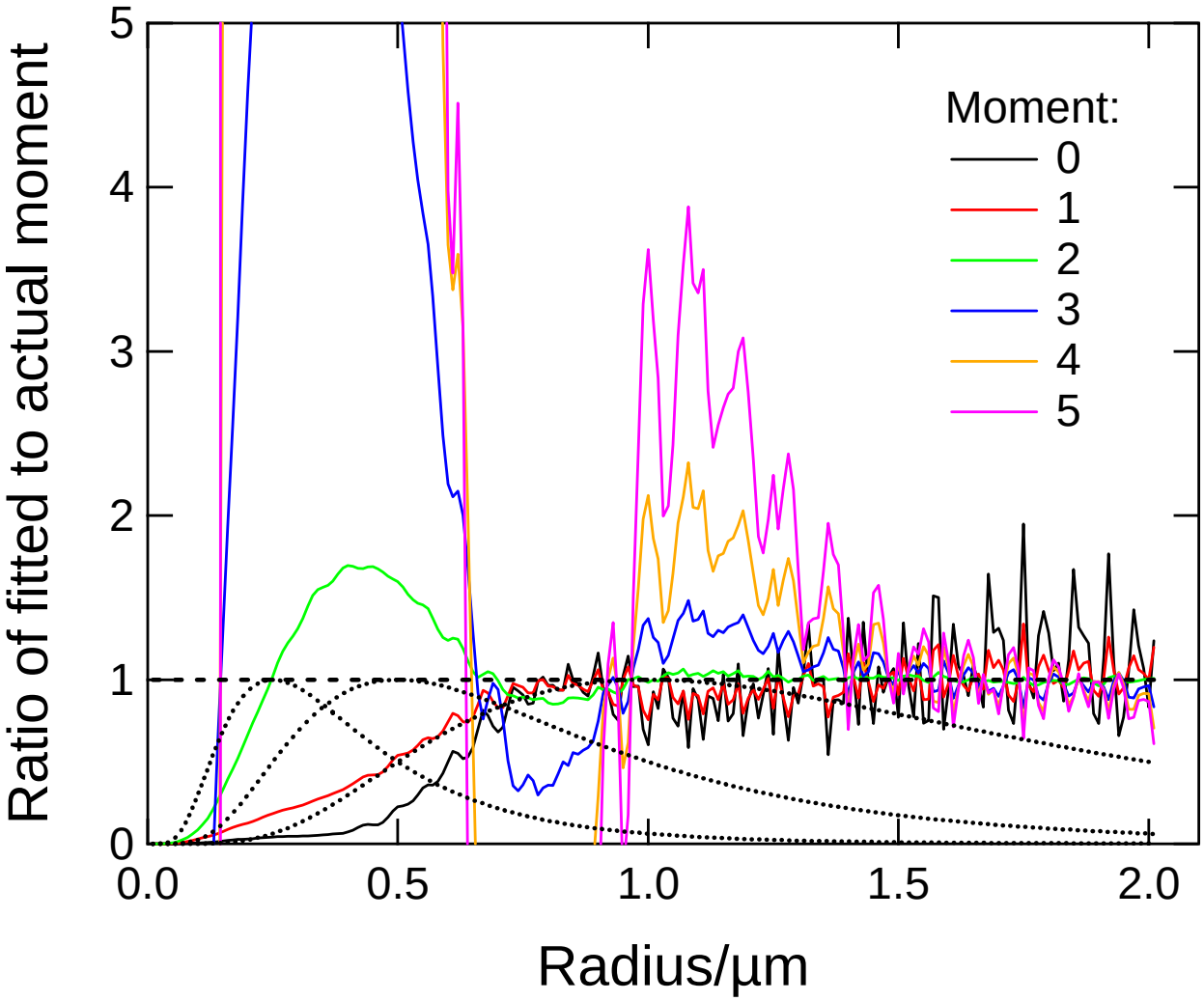
MOMENT COMPARISONS

For lognormal distributions with $\sigma = 1.8$, $N_0 = 1$

r_g/μm	0.25		0.5		1.0	
p	**Act**	**Fit6**	**Act**	**Fit6**	**Act**	**Fit6**
0	0.44	0.096	0.84	0.49	1.27	1.11
1	0.18	0.10	0.65	0.54	1.48	1.41
2	0.11	0.12	0.63	0.66	1.97	1.98
3	0.081	0.17	0.74	0.89	2.86	2.99
4	0.079	0.25	0.98	1.29	4.42	4.78
5	0.092	0.40	1.42	1.96	7.15	7.92

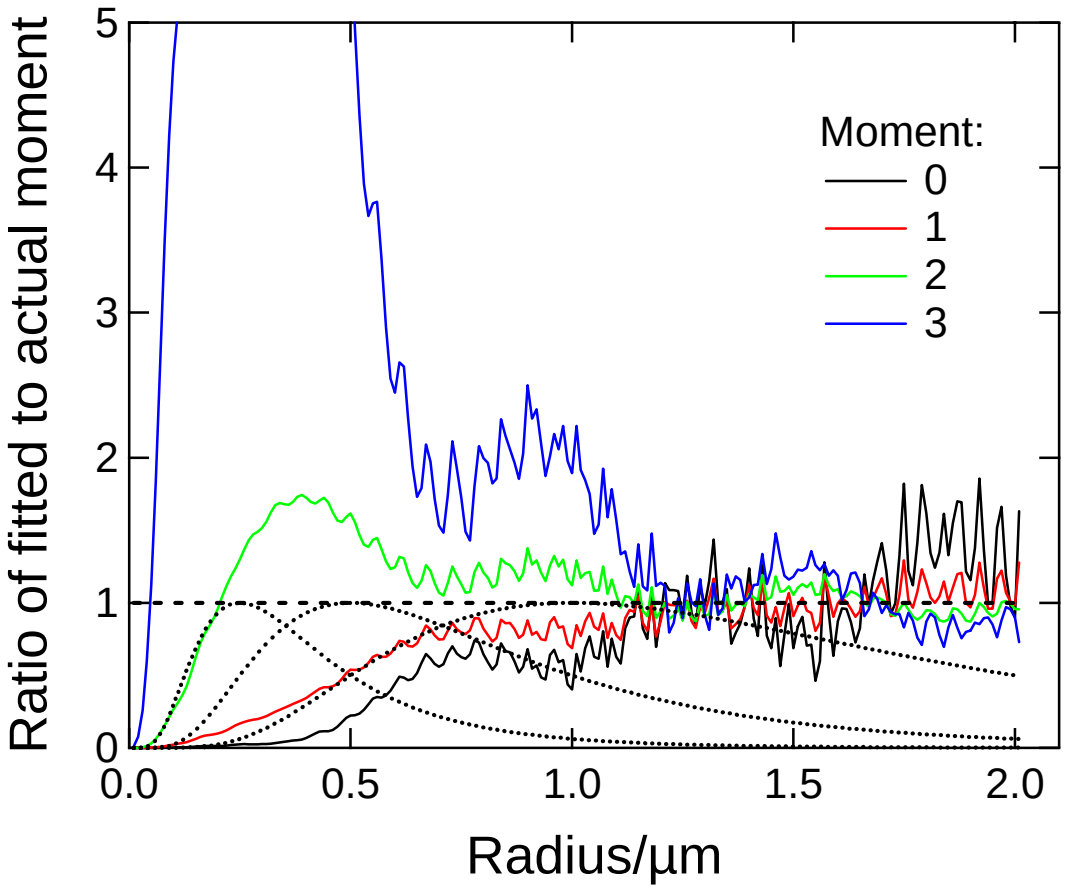
Red denotes agreement to within 25%.

RESULTS DEPEND ON SIZE DISTRIBUTION



RECONSTRUCTION OF MOMENTS WITH TWO WAVELENGTHS

$r < 2$ μm, $m = 1.40$ (~ 70% RH), λ = 450, ~~550~~, 700 μm



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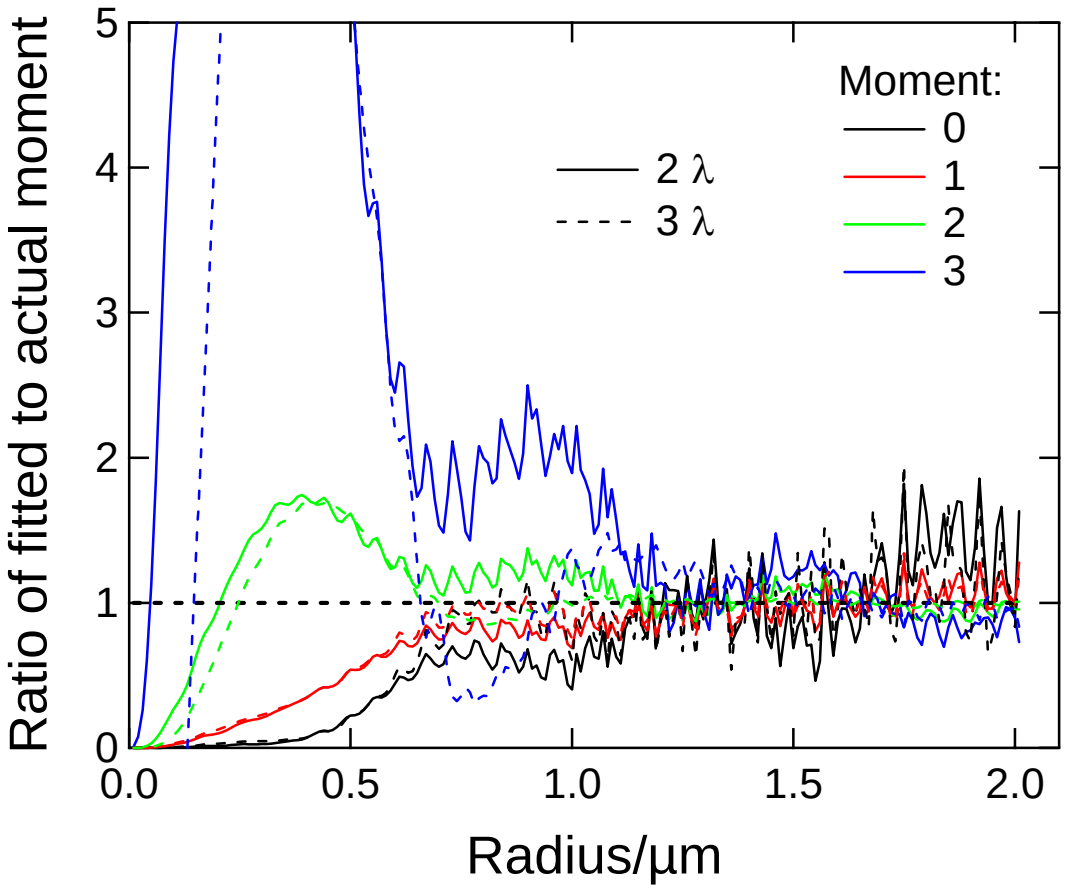

MOMENT COMPARISONS - 2 λs

For lognormal distributions with $\sigma = 1.8$, $N_0 = 1$

r_g/μm	0.25			0.5			1.0		
p	**Act**	**Fit6**	**Fit4**	**Act**	**Fit6**	**Fit4**	**Act**	**Fit6**	**Fit4**
0	0.44	0.096	0.077	0.84	0.49	0.49	1.27	1.11	1.01
1	0.18	0.10	0.094	0.65	0.54	0.54	1.48	1.41	1.37
2	0.11	0.12	0.013	0.63	0.66	0.66	1.97	1.98	2.04
3	0.081	0.17	0.21	0.74	0.89	0.89	2.86	2.99	3.24
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5	0.092	0.40		1.42	1.96		7.15	7.92	

Red denotes agreement to within 25%.

SUMMARY

Nephelometer measurements can provide information on the lowest several moments of the aerosol size distribution.

Further work will investigate:

- Effect of index of refraction
- More realistic *BSF*, nephelometer measurements
- Smoothing with inversion

Questions:

- How many wavelengths are required?
- How much information would be gained/lost by more/fewer λ?
- Which wavelengths are optimal?
  - (probably as far apart as possible)
- Can modifications be made to yield other useful measurements?
  - (such as conditioning or choice of RH)